Numerical study of the transmission of energy in discrete arrays of sine-Gordon equations in two space dimensions

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(Received 4 June 2007; revised manuscript received 25 September 2007; published 18 January 2008)

In this paper, we provide a numerical approximation to the occurrence of the process of nonlinear supratransmission in semiunbounded, discrete, (2+1)-dimensional systems of sine-Gordon equations subject to harmonic Neumann boundary data irradiating with a frequency in the forbidden band gap. The model is a generalization of the one describing semi-infinite, discrete, (1+1)-dimensional, parallel arrays of Josephson junctions connected through superconducting wires, subject to the action of an ac current at the end. The computational results are obtained using a finite-difference scheme for sine-Gordon and nonlinear Klein-Gordon media, and the method is applied to systems of harmonic oscillators when Dirichlet data are imposed to the boundary. Our numerical results show that energy is transmitted into the medium in the form of discrete breathers.

DOI: 10.1103/PhysRevE.77.016602 PACS number(s): 05.45.Yv, 02.60.Lj, 63.20.Pw

I. INTRODUCTION

When one talks about the analytical study of the process of supratransmission in nonlinear media, one must necessarily talk about the pioneering work of Geniet and Leon [1] in coupled systems of sine-Gordon equations. The problem—which consists in the determination of the critical amplitude at which a nonlinear medium subject to a harmonic disturbance irradiating at a frequency in the forbidden band gap propagates nonlinear intrinsic modes—is characterized by a sudden increase in the energy injected in the system by the driving boundary.

Supratransmission has been studied in many one-dimensional discrete systems like the Fermi-Pasta-Ulam model [2], systems of sine-Gordon and Klein-Gordon equations [1], double sine-Gordon equations [3], Bragg media in the nonlinear Kerr regime [4], and even in continuous media governed by sine-Gordon equations subject to Dirichlet [5] or Neumann boundary data [6]. The presence of the process of supratransmision in discrete systems with two space dimensions has been suggested for the case of Neumann boundary data [6], and several application papers have been realized in the one-dimensional case [7–10].

Analytically, the process of nonlinear supratransmission has been studied in the discrete Fermi-Pasta-Ulam [2] and sine-Gordon [5,6] models through the continuous-limiting case when the coupling coefficient is large. In those cases, the quoted citations possess accurate mathematical predictions of the phenomenon. However, we must emphasize that the two-dimensional scenario has been left aside by the specialized literature, mainly due to the fact that the analytical study of this situation is more complicated than its one-dimensional counterpart; needless to mention that a three-dimensional version of the problem is, by far, more challenging [11].

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From a pragmatical perspective and in view of the characterization of supratransmission in the energy domain, a thorough numerical analysis of the process should make use of techniques that provide consistent approximations of the problem under study both in the solution domain and in the energy domain. An analysis like this has been satisfactorily carried out, for instance, in [12], or in the numerical study of propagation of binary signals in media governed by coupled sine-Gordon equations [13,14].

In the present paper, we study numerically the process of supratransmission in semiunbounded, discrete, two-dimensional media governed by coupled sine-Gordon equations when the boundary is subject to harmonic driving. Our computations are based on a numerical method with suitable computational characteristics to analyze the process, which in turn is a generalization of a method for conservative systems [15]. We must remark that the results presented in this work have been checked in the solution domain against a classical Runge-Kutta method of order 4; however, we must clarify that the simulations presented here make use of the finite-difference scheme introduced in Sec. III.

II. MATHEMATICAL MODEL

Our point of departure is motivated by the study of semi-infinite, discrete arrays of parallel Josephson junctions connected through superconducting wires and perturbed harmonically at the boundary by a frequency in the forbidden band gap [6,16,17]. In this context, the (1+1)-dimensional model studied is

$$\ddot{u}_n - c^2 \Delta_x u_n + \gamma \dot{u}_n + \sin u_n = 0, \quad n \in \mathbb{Z}^+, \tag{1}$$

subject to boundary condition $c^2(u_0 - u_1) = \phi$, where $\phi(t) = A \sin(\Omega t)$. Here, the real variable u_n physically represents the gauge invariant phase difference [16], and $\Delta_x u_n$ denotes the finite second difference $u_{n+1} - 2u_n + u_{n-1}$. Both the

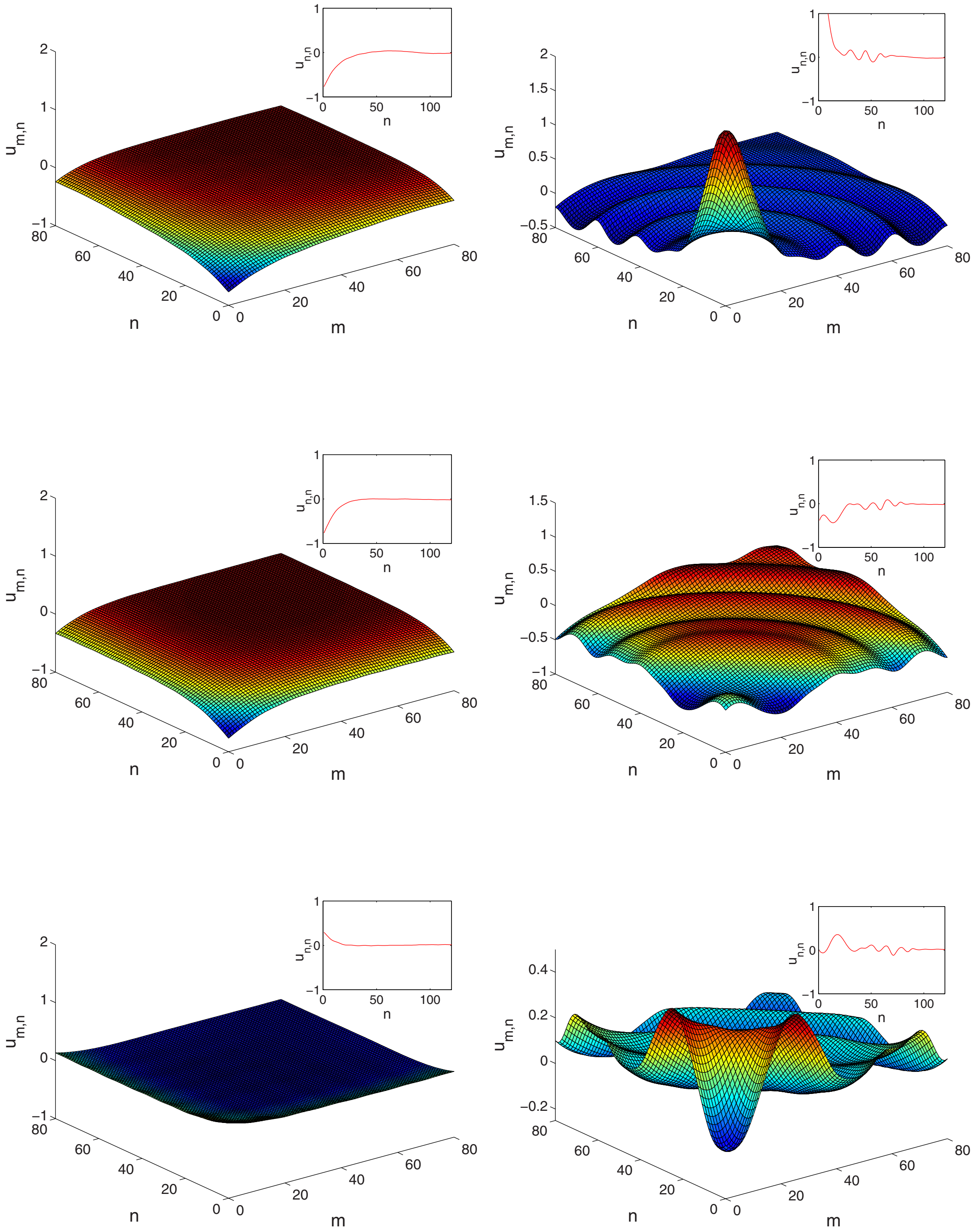

FIG. 1. (Color online) Snapshots of solutions to undamped problem (3) with $c=5$ at three different times ($t=139$ for the first row, $t=144$ for the second row, and $t=147$ for the third row), for a driving frequency of 0.9 and two driving amplitudes: one at which supratransmission has not taken place ($A=1.05$, left-hand column), and another at which it has just started ($A=1.06$, right-hand column). The inset figures represent the solutions of the problem on the diagonal line $m=n$. The graphs are presented as numerical evidence of the existence of a critical threshold above which transmission of energy in (3) takes place.

coupling coefficient c and the damping coefficient are assumed to be non-negative numbers, and $\Omega < 1$.

In order to extend this problem to a higher dimensional scenario, we let $u_{m,n}$ be a real function of time for every pair of non-negative integers m and n. Define the discrete Laplacian operator

$$\Delta u_{m,n} = u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n}, \quad (2)$$

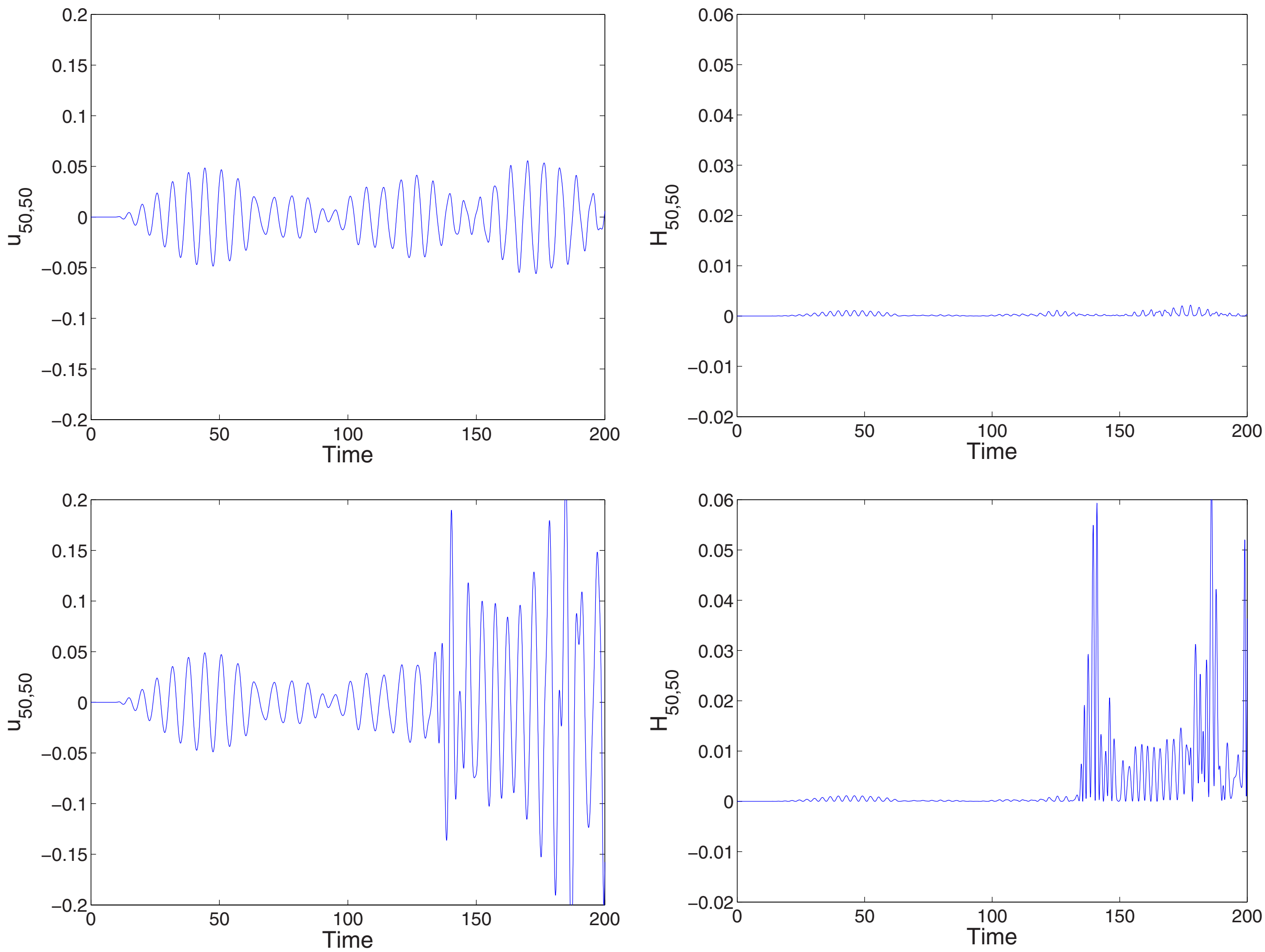

FIG. 2. (Color online) Graphs of state versus time (top row) and local energy versus time (bottom row) of the junction located in node (50,50) in an array consisting of 150^2 nodes governed by (3), subject to a harmonic driving with a frequency of 0.9, and an amplitude of 1.05 (left-hand column) and 1.06 (right-hand column). A potential function $G(u)=1-\cos u$ and a coefficient of coupling equal to 5 were employed. The figure is presented as a preliminary proof of the existence of supratransmission in two-dimensional discrete arrays described by (3).

for every pair of positive integers m and n. For such values of m and n, this work studies the problem

$$\ddot{u}_{m,n} - c^2\Delta u_{m,n} + \gamma\dot{u}_{m,n} + G'(u_{m,n}) = 0,$$
$$\text{s.t.} \begin{cases} u_{m,n}(0) = \dot{u}_{m,n}(0) = 0, \\ u_{m,0} - u_{m,1} = u_{0,n} - u_{1,n} = \phi/c^2. \end{cases} \quad (3)$$

By analogy with the one-dimensional case, c is called the coupling coefficient and γ is the coefficient of external damping, while G is either the classical potential for a sine-Gordon system—when $G(u)=1-\cos u$—or the potential for a nonlinear Klein-Gordon equation—in the case that $G(u)=\frac{1}{2!}u^2-\frac{1}{4!}u^4+\frac{1}{6!}u^6$. Meanwhile, the function ϕ is the harmonic function defined above, which irradiates at a frequency Ω in the forbidden band gap of the medium, that is, $\Omega<1$.

It is worth noticing that the conservative case yields a Hamiltonian for the junction in node (m,n) given by the expression

$$H_{m,n} = \frac{1}{2}\dot{u}_{m,n}^2 + \frac{c^2}{2}(u_{m+1,n}-u_{m,n})^2 + \frac{c^2}{2}(u_{m,n+1}-u_{m,n})^2 + G(u_{m,n}), \quad (4)$$

for any positive integers m and n. After including the potential functions between the couplings in the boundaries of the system, it can be checked that the total energy of the medium at a fixed time is provided by

$$E = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} H_{m,n} + \frac{c^2}{2}\left(\sum_{m=1}^{\infty}(u_{m,1}-u_{m,0})^2 + \sum_{n=1}^{\infty}(u_{1,n}-u_{0,n})^2\right). \quad (5)$$

III. NUMERICAL TECHNIQUE

Take a regular partition $0=t_0<t_1<\cdots<t_M=T$ of the time interval $[0,T]$ with time step equal to Δt, and for each

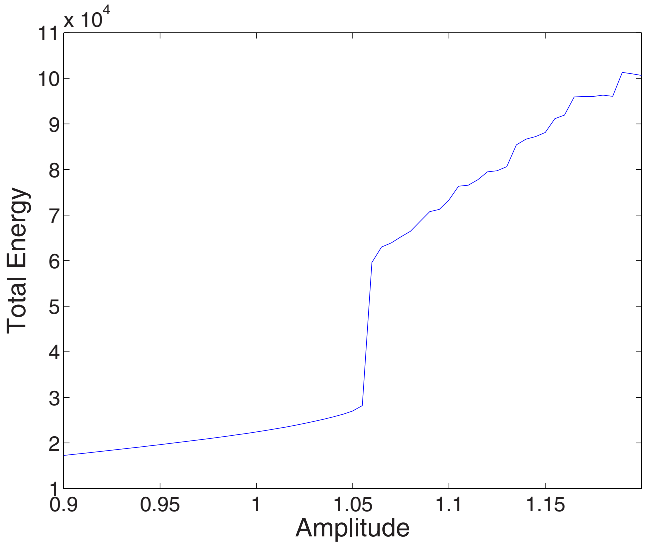

FIG. 3. (Color online) Graph of total energy of a system governed by (3) versus driving amplitude, for a driving frequency of 0.9, a period of time equal to 200, a potential function $G(u)=1-\cos u$, and $c=5$.

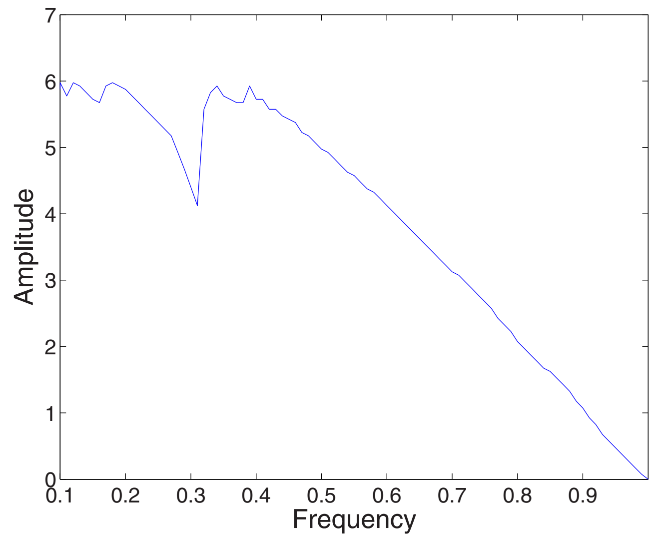

FIG. 4. (Color online) Diagram of bifurcation of smallest driving amplitude at which nonlinear supratransmission starts versus driving frequency, for an undamped large medium governed by (3), with $G(u)=1-\cos u$ and $c=5$.

$k=0,1,\ldots,M$, represent the approximate solution to our problem on site (m,n) at time t_k and the value of the function ϕ at t_k by $u_{m,n}^k$ and ϕ_k, respectively. Conveying that

$$\delta_t u_{m,n}^k = u_{m,n}^{k+1} - u_{m,n}^{k-1},$$

$$\delta_t^2 u_{m,n}^k = u_{m,n}^{k+1} - 2u_{m,n}^k + u_{m,n}^{k-1},$$

$$\delta_x^2 u_{m,n}^k = u_{m+1,n}^k - 2u_{m,n}^k + u_{m-1,n}^k,$$

$$\delta_y^2 u_{m,n}^k = u_{m,n+1}^k - 2u_{m,n}^k + u_{m,n-1}^k,$$

the problem under study takes the following discrete form, for every positive integer m and n:

$$\frac{\delta_t^2 u_{m,n}^k}{(\Delta t)^2} - c^2(\delta_x^2 u_{m,n}^k + \delta_y^2 u_{m,n}^k) + \frac{\gamma}{2\Delta t}\delta_t u_{m,n}^k + \frac{G(u_{m,n}^{k+1}) - G(u_{m,n}^{k-1})}{u_{m,n}^{k+1} - u_{m,n}^{k-1}} = 0,$$

$$\text{s.t.} \begin{cases} u_{m,n}^0 = u_{m,n}^1 = 0, \\ c^2(u_{m,0}^k - u_{m,1}^k) = \phi_k, \\ c^2(u_{0,n}^k - u_{1,n}^k) = \phi_k. \end{cases} \tag{6}$$

The local energy $H_{m,n}$ of the Josephson junction located at (m,n) in the kth time step will be approximated numerically by the discrete expression

$$H_{m,n}^k = \frac{1}{2}\left(\frac{u_{m,n}^{k+1} - u_{m,n}^k}{\Delta t}\right)^2 + \frac{c^2}{2}(u_{m+1,n}^{k+1} - u_{m,n}^{k+1})(u_{m+1,n}^k - u_{m,n}^k) + \frac{c^2}{2}(u_{m,n+1}^{k+1} - u_{m,n}^{k+1})(u_{m,n+1}^k - u_{m,n}^k) + \frac{G(u_{m,n}^{k+1}) + G(u_{m,n}^k)}{2}. \tag{7}$$

Meanwhile the total energy at time t_k will be approximated by the formula

$$E_k = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} H_{m,n}^k + \frac{c^2}{2}\left(\sum_{m=1}^{\infty}(u_{m,1}^{k+1} - u_{m,0}^{k+1})(u_{m,1}^k - u_{m,0}^k) + \sum_{n=1}^{\infty}(u_{1,n}^{k+1} - u_{0,n}^{k+1})(u_{1,n}^k - u_{0,n}^k)\right). \tag{8}$$

It is worth noticing that the computational schemes presented in this section are consistent with the problem under study in this paper, both in the solution and the energy domains. Moreover, the method is conditionally stable, having a stability region given by $8c^2(\Delta t)^2 - \gamma\Delta t < 4$. Technically, the method is implicit and nonlinear, so that its practical implementation requires Newton's method to solve systems of nonlinear equations. It is not difficult to check that this procedure yields a tridiagonal system which may be solved by Crout's technique for band matrices.

IV. SIMULATIONS

We employ the numerical method described in the preceding section in order to approximate solutions of (3). Numerically, we will fix $c=5$, $\gamma=0$, a time step of 0.05, and consider a square system of $N\times N$ nodes satisfying (3) and the boundary conditions

$$u_{m,N+1} - u_{m,N} = 0,$$

$$u_{N+1,n} - u_{N,n} = 0, \tag{9}$$

where $N=150$. Moreover, in order to simulate an unbounded medium we include a variable damping coefficient $\gamma_{m,n}$ in each node (m,n), which is equal to zero everywhere except in those nodes satisfying $m \geq 120$ or $n \geq 120$, in which case $\gamma_{m,n}=5$. Furthermore, observe that the boundary conditions (9) translate into the discrete expressions

$$u^k_{m,N+1} - u^k_{m,N} = 0,$$

$$u^k_{N+1,n} - u^k_{N,n} = 0. \tag{10}$$

In addition, in order to avoid the generation of shock waves at the initial time, we slowly and linearly increase the driving amplitude at the boundary during a finite period of time before reaching the constant desired amplitude.

To start with, we fix the driving frequency of the medium at $\Omega=0.9$, and compute the solution of the system for three different times and for two different driving amplitudes, $A=1.05$ and $A=1.06$. The results are presented in Fig. 1, in which the left-hand column shows the states of the system corresponding to $A=1.05$ for times $t=139,144,147$, while the right-hand column shows the corresponding solutions for $A=1.06$. It is clear that the propagation of energy into the system takes place in the latter case (apparently in the form of solitary waves). This fact is verified by the inset figures associated to each graph, in which the corresponding solutions for the nodes in the diagonal $m=n$ are presented.

The time evolution of the junction on node (50,50) over a time period of 200 is checked then in the solution and the energy domains, obtaining thus Fig. 2. Observe that the case when the amplitude is 1.05 yields a local energy which is approximately equal to zero at all time. On the other hand, when the amplitude is equal to 1.06, a drastic increase in the local energy is observed around the time 140. This results evidence numerically the existence of supratransmission in system (3), at least for $\Omega=0.9$.

Next, we obtain the total energy E of the medium under study for several values of A and a fixed frequency of 0.9. In view of the evidence introduced in the previous paragraph, we expect to see a drastic increase in the total energy of the

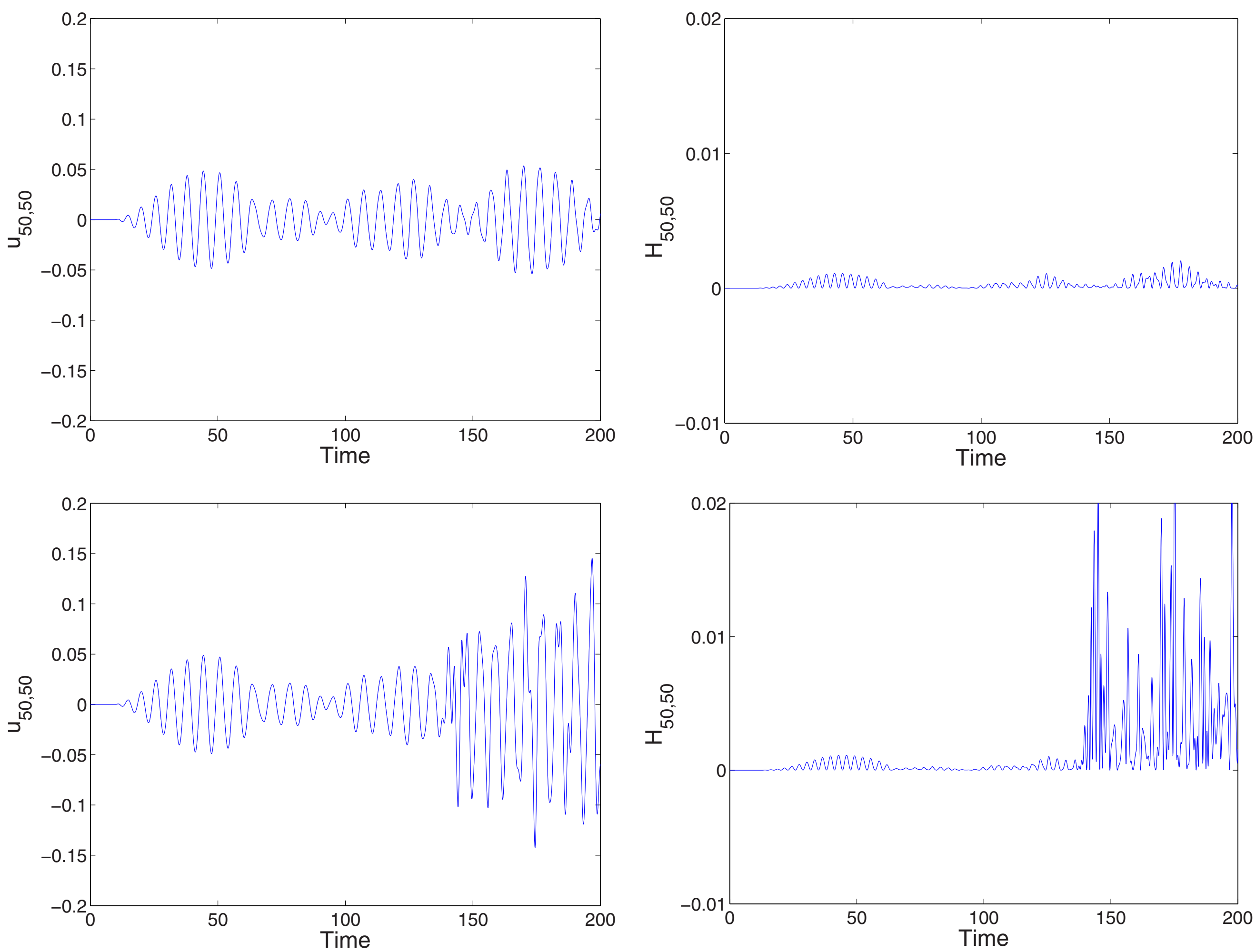

FIG. 5. (Color online) Graphs of state versus time (top row) and local energy versus time (bottom row) of the junction located in node (50,50) in an array consisting of 150^2 nodes governed by (3), subject to a harmonic driving with a frequency of 0.9, and an amplitude of 1.05 (left-hand column) and 1.06 (right-hand column). A potential function $G(u)=\frac{1}{2!}u^2-\frac{1}{4!}u^4+\frac{1}{6!}u^6$ and a coefficient of coupling equal to 5 were employed.

system around the value $A=1.06$. The behavior of E versus A is depicted in Fig. 3, for a period of time equal to 200. The results clearly confirm the presence of the process of nonlinear supratransmission in our medium. Figure 4 gives an estimate of the critical amplitude at which supratransmission starts versus driving frequency.

Finally, we wish to establish that supratransmission is likewise present in $(2+1)$-dimensional systems consisting of nonlinear Klein-Gordon equations. First, we obtain graphs of time evolution of the amplitude and the local energy of node (50,50), for $\Omega=0.9$ and two different values of A, 1.05 and 1.06. The results are shown in Fig. 5, and they establish that there exists a drastic change in the energy domain between the values of amplitude employed.

We study next the behavior of the total energy of the system over a time period of 200 versus driving amplitude, for the same fixed driving frequency. The results are displayed in Fig. 6; it clearly establishes the presence of supratransmission in this situation.

V. MECHANICAL SYSTEMS

Consider a semi-infinite chain of identical harmonic oscillators initially at rest in their natural positions, coupled through identical springs with a constant coupling coefficient c. The first pendulum of the chain is perturbed harmonically with a frequency Ω in the forbidden band gap of the system, and the medium is assumed to suffer the actions of constant external damping. The model describing the physical behavior of this system is given by the mixed-value problem

$$\frac{d^2u_n}{dt^2} - c^2\Delta_x u_n + \gamma\frac{du_n}{dt} + G'(u_n) = 0,$$

$$\text{s.t.}\begin{cases} u_n(0) = 0, & n \in \mathbb{Z}^+, \\ \dfrac{du_n}{dt}(0) = 0, & n \in \mathbb{Z}^+, \\ u_0(t) = \phi(t), & t \geq 0. \end{cases} \tag{11}$$

Assume that $u_{m,n}$ is a real function of time for every $m,n \in \mathbb{Z}^+$. A natural generalization of mixed-value problem (11) to two space dimensions is given by the initial-value problem with Dirichlet boundary data

$$\ddot{u}_{m,n} - c^2\Delta u_{m,n} + \gamma\dot{u}_{m,n} + G'(u_{m,n}) = 0,$$

$$\text{s.t.}\begin{cases} u_{m,n}(0) = \dot{u}_{m,n}(0) = 0, \\ u_{m,0}(t) = u_{0,n}(t) = \phi(t). \end{cases} \tag{12}$$

For our computations, the boundary conditions will take the form $u^k_{m,0}=u^k_{0,n}=\phi_k$. We choose a coupling coefficient equal to 5, a time step of 0.05, $N=300$, and we drive system (12) with $\gamma=0.01$ and $\mathbf{m}=0$ at a frequency $\Omega=0.9$. Two amplitude values are employed, 1.52 and 1.53. The resulting simulations at three different times are shown in Fig. 7. The graphs evidence a drastic change in the behavior of solutions; in fact, for $A=1.52$ there is no evidence of transmission of energy into the system by the driving boundary. Meanwhile, for $A=1.53$ our results show that nonlinear modes (discrete breathers, to be exact) are created at the origin.

We look at these circumstances locally at a specific site, in the solution and the energy domains. Indeed, Fig. 8 presents graphs of solution $u_{50,50}$ and local energy $H_{50,50}$ versus time of site (50,50) in a system (12) driven with the specifications quoted in the paragraph above. Again, a well-defined difference in the behavior of solutions is perceived between the driving amplitude values 1.52 and 1.53, indicating thus the presence of supratransmission in the medium. A similar situation happens in the energy domain: whereas the energy is essentially equal to zero for $A=1.52$, it is evidently nonzero for $A=1.53$. Evidently, the graphs of $u_{50,50}$ and $H_{50,50}$ versus time are consistent in both cases. Moreover, the qualitative behavior of the solution when $A=1.53$ resembles qualitatively the $(1+1)$-dimensional case [1].

Next, we consider driving frequencies ranging on the interval $[0.1,1]$. For each such frequency and for driving amplitudes varying in $[0,10]$, we compute the total energy of the system during a time period of $T=2000$ by integrating numerically the function of total energy over $[0,T]$. The results (not presented here for the sake of briefness) establish a sudden increase in the total energy of the system at a certain critical amplitude that we call the *supratransmission threshold*. The graph of approximate supratransmission threshold versus driving frequency of a system (12) with coupling coefficient equal to 5 is presented in Fig. 9, using the standard methodology [1,13]. Here we must notice that the supratransmission threshold is a decreasing function of the driving amplitude on $[0.5,1]$, and that the phenomenon of harmonic phonon quenching is present. This is in agreement with the $(1+1)$-dimensional case.

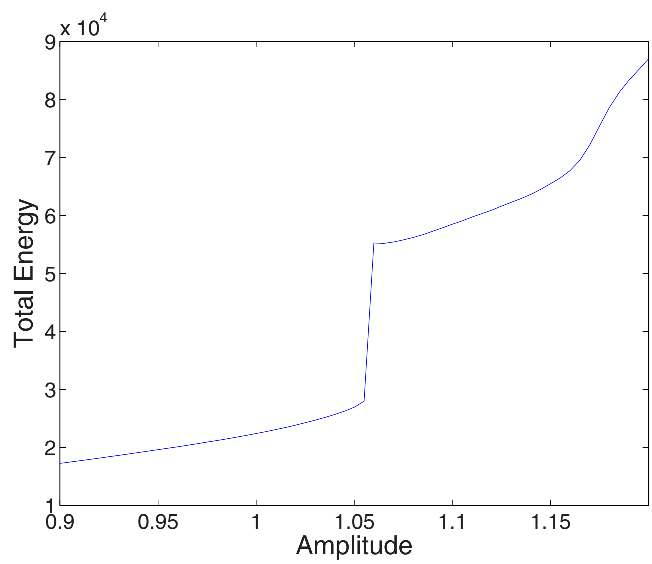

FIG. 6. (Color online) Graph of total energy of a system governed by (3) versus driving amplitude, for a driving frequency of 0.9, a period of time equal to 200, $G(u)=\frac{1}{2!}u^2-\frac{1}{4!}u^4+\frac{1}{6!}u^6$, and $c=5$.

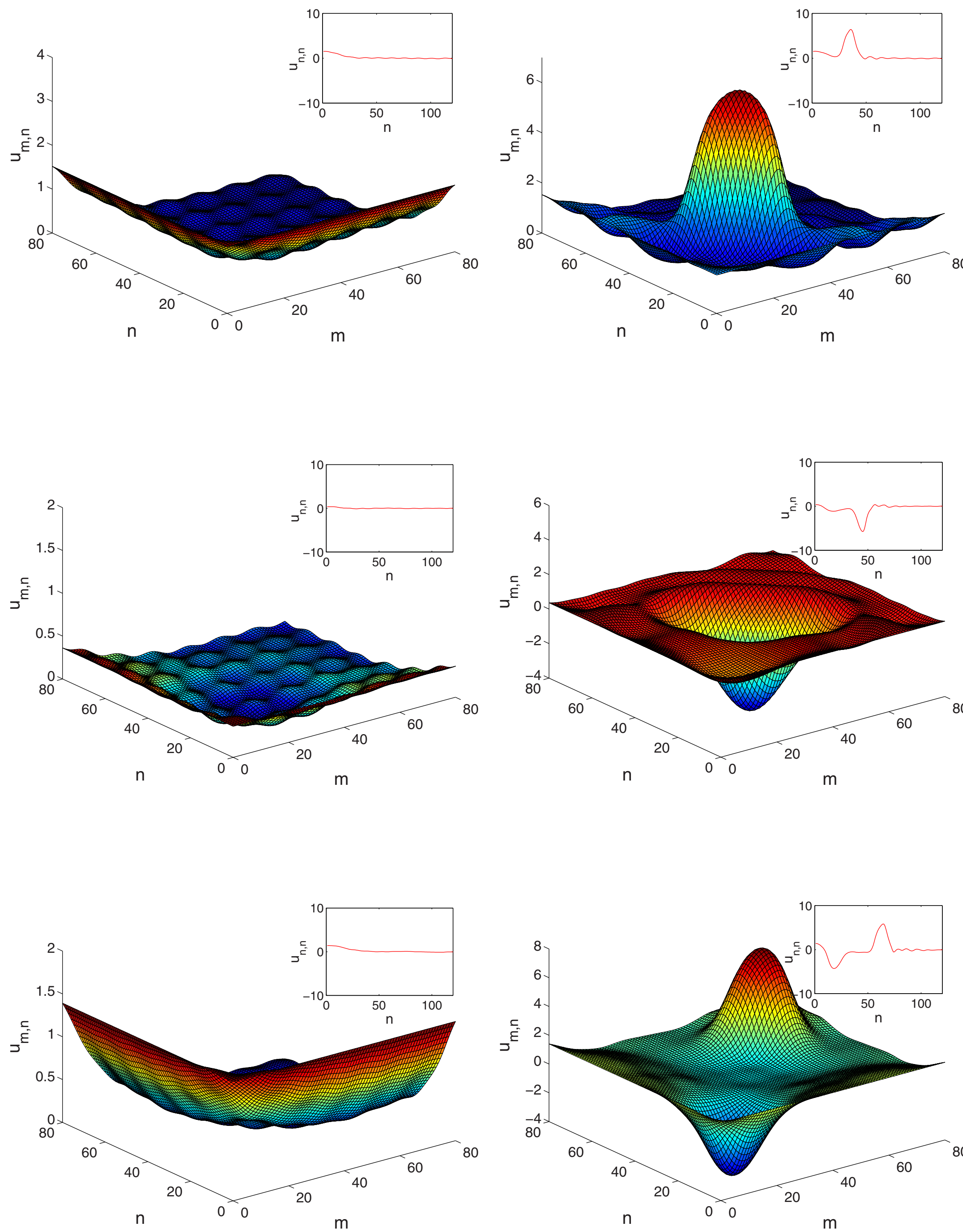

FIG. 7. (Color online) Snapshots of solutions to undamped problem (12) with $c=5$ at three different times ($t=113$ for the first row, $t=119$ for the second row, and $t=127$ for the last row), for a driving frequency of 0.9 and two driving amplitudes: one at which supratransmission has not taken place ($A=1.52$, left-hand column), and another at which it has just started ($A=1.53$, right-hand column). The inset figures represent the solutions of the problem on the diagonal line $m=n$. The graphs are presented as numerical evidence of the existence of a critical threshold above which transmission of energy in (12) takes place.

VI. CONCLUSIONS AND PERSPECTIVES

In this work, we have presented numerical evidence that the processes of supratranmission is present in two-dimensional systems of bounded discrete arrays that generalize the model describing Josephson junctions coupled through superconducting wires and subject to harmonic disturbances in two adjacent sides. Bifurcation diagrams estab-

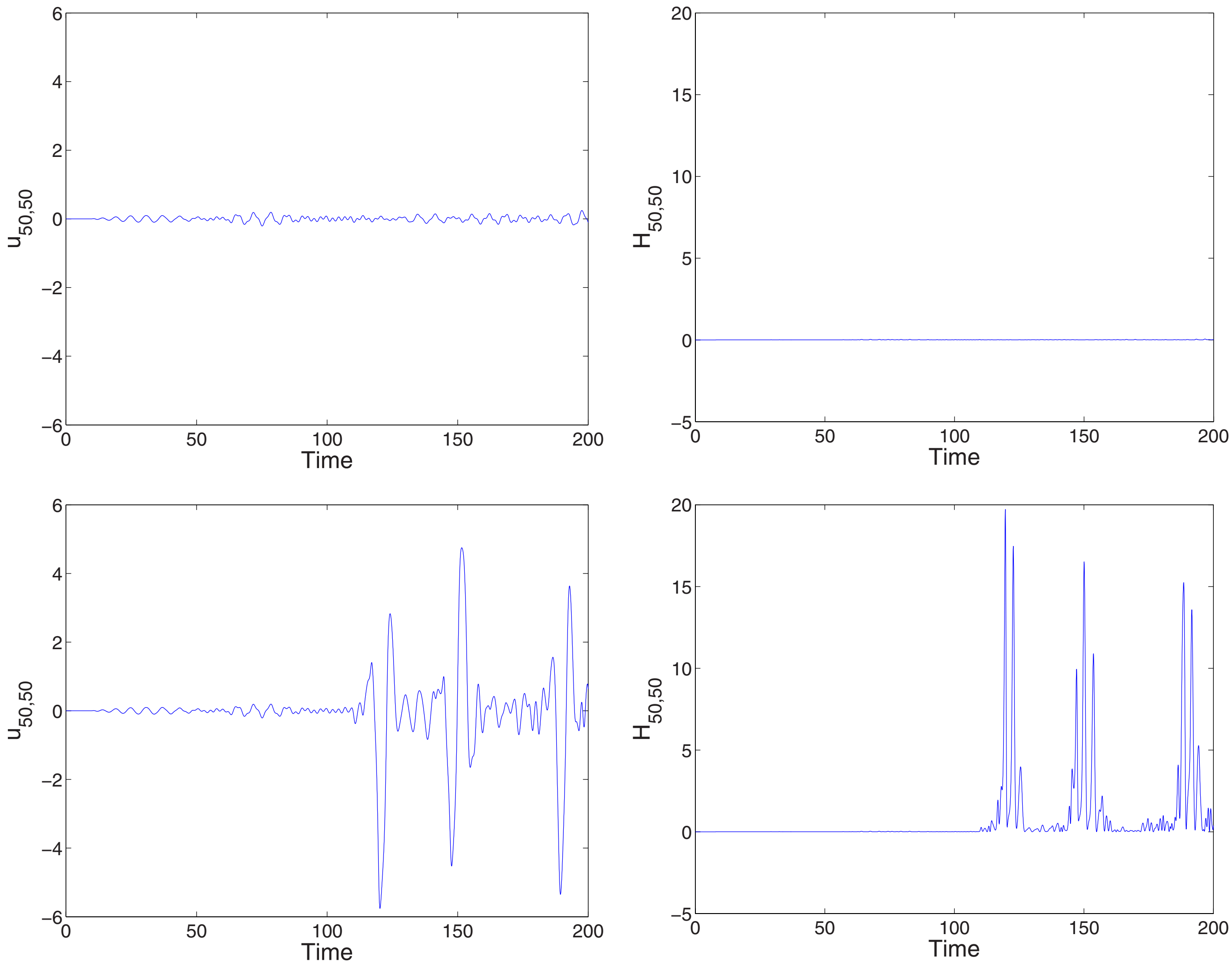

FIG. 8. (Color online) Time-dependent graphs of approximate solutions $u_{50,50}$ and local energies $H_{50,50}$ of the junction located on site (50,50) for problem (12), for a driving frequency of 0.9 and two driving amplitudes: one at which supratransmission has not taken place (left-hand column), and another at which it has just started (right-hand column).

lishing the relationship between the critical amplitude at which supratransmission first occurs versus driving frequency have been provided with the help of a computational technique with consistency properties in the solution and energy domains.

The computational results have been validated by the use of traditional computational techniques. Moreover, our work has established that the process of nonlinear supratransmission is also present in two-dimensional systems described by discrete nonlinear Klein-Gordon equations, and even in systems of sine-Gordon equations subject to Dirichlet boundary data, which is a model that arises in the physical description of lattices consisting of harmonic oscillators coupled through identical springs.

Several questions meriting attention are still left open. For instance, the discovery of the analytical apparatus prescribing the dependence of the critical amplitude with respect to the driving frequency, at least for the continuous limit case in which the coupling coefficient tends to infinity, is a matter of the utmost importance. Moreover, in view of the fact that

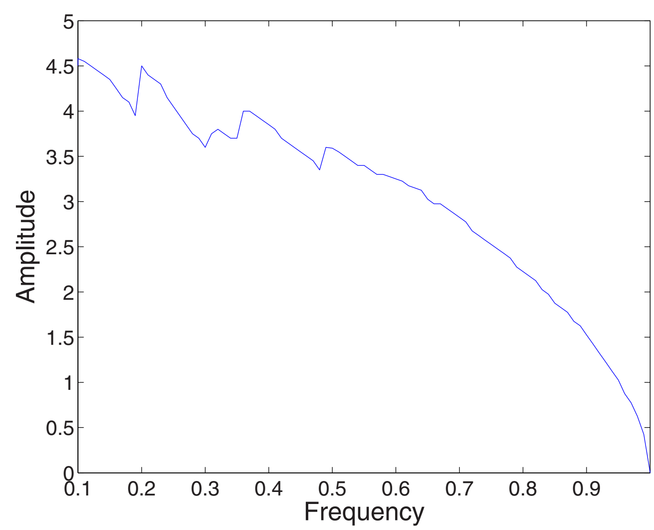

FIG. 9. (Color online) Graph of critical amplitude at which supratransmission starts versus driving frequency for problem (12).

one-dimensional and two-dimensional systems of coupled sine-Gordon or Klein-Gordon equations have the capacity of transmitting energy in the forbidden band gap, the question arises as if higher dimensional systems possess the same capability.

On the other hand, it has been proved that supratransmission survives in the presence of external damping [3], and the inclusion of several other parameters such as internal damping, relativistic mass, and generalized Josephson currents. The process of nonlinear supratransmission is also expected to be present in the $(2+1)$-dimensional scenario under the presence of the same parameters, in which case, the study of the effects of those parameters in the occurrence of supratransmission is a task that merits further research. *A priori*, one expects to obtain similar qualitative results as the ones obtained in the one-dimensional situation.

ACKNOWLEDGMENTS

The author wishes to thank the anonymous referee for their invaluable comments. Also, the author wishes to acknowledge support from Dr. Álvarez Rodríguez, Dean of the Faculty of Sciences of the Universidad Autónoma de Aguascalientes, and from Dr. Avelar González, head of the Office for Research and Graduate Studies of the same university. The present work represents a set of partial results under project PIM08-1 at Universidad Autónoma de Aguascalientes.

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